

Constraints

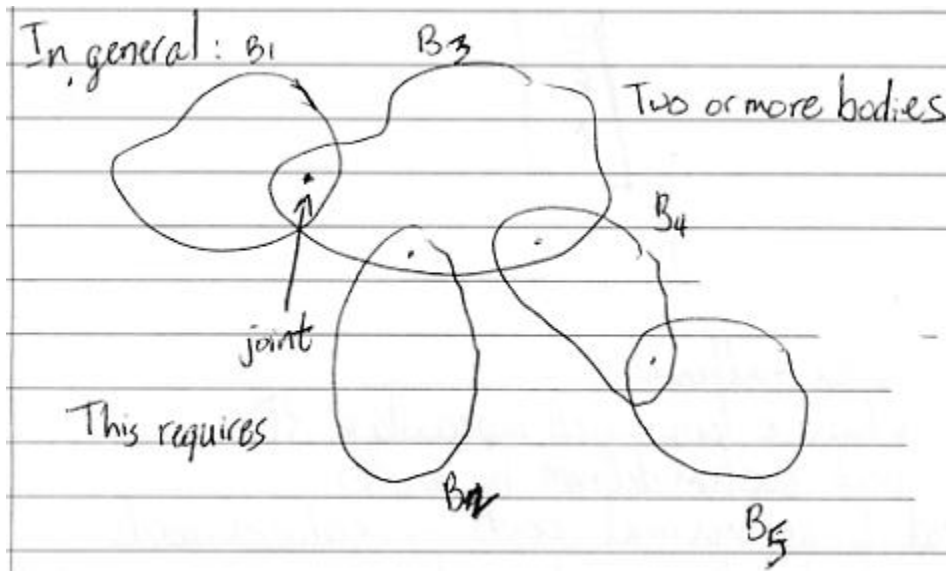
Examples of constraints:

Hinge joints, anatomical joints like a shoulder, elbow, or wrist.

Different joints have different degrees of freedom and have different limitations on that freedom.

Consider a model with two or more bodies connected by joints. The bodies can also be represented in an acyclic graph or in more complicated hierarchies a cyclic one.

Diagram 1:



Graph 1:

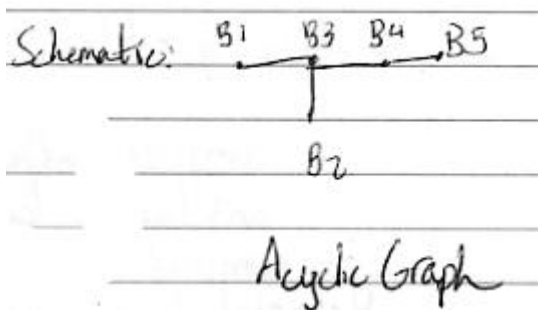
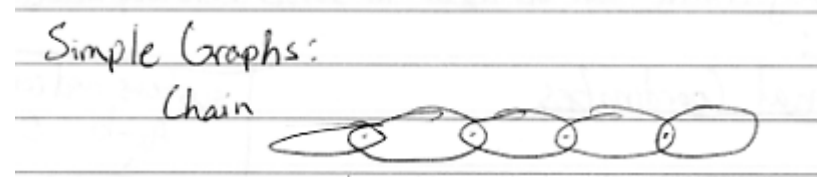


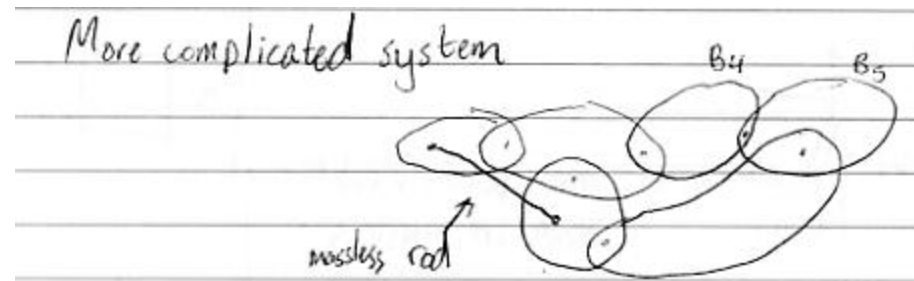
Diagram 2:



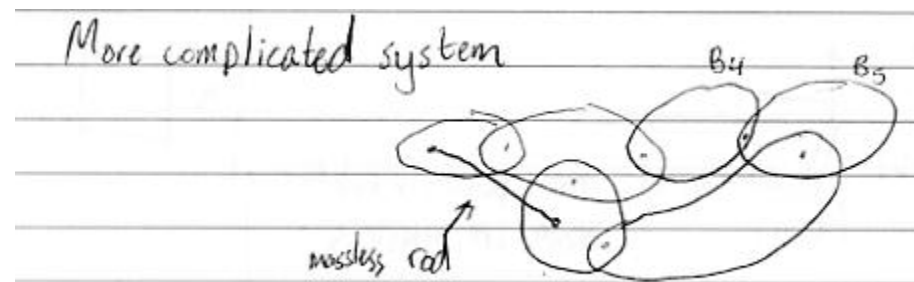
Chain system

Diagram 3:

More complicated system with a loop.



Graph 3:

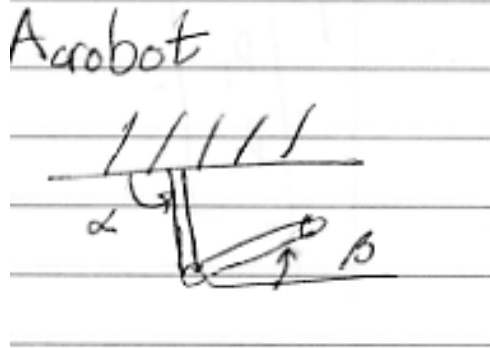


From last lecture regarding Lagrange Dynamics

$$M(q) * q'' = f(q, q', t)$$

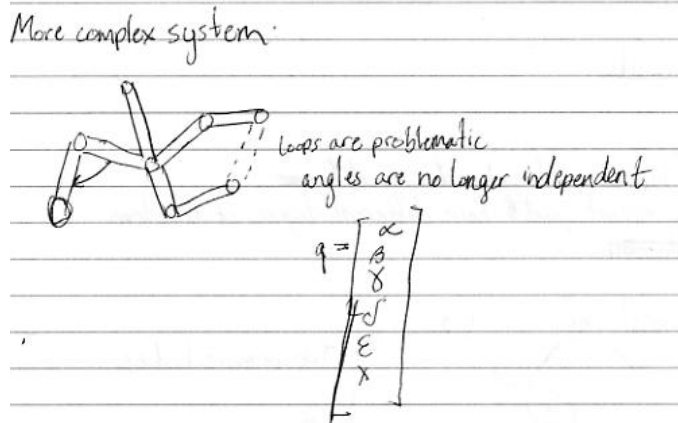
$$Q = [\alpha, \beta]$$

Acrobot Diagram 1:



In this situation $M(q) * q'' = f(q, q', t)$ but it is much more complex.

Acrobot Diagram 2:



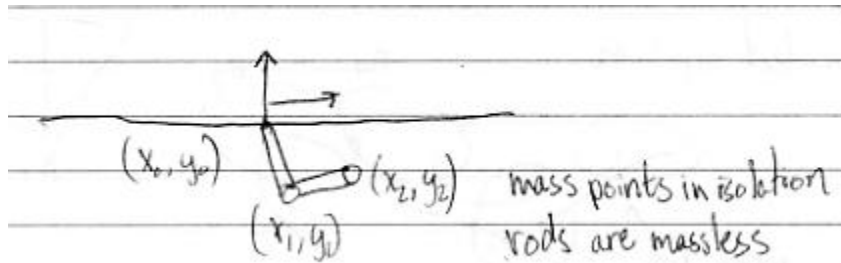
This is referred to as the **minimal coordinate** or **reduced coordinate** approach.

Features of this approach:

- Complex mathematics
- Can't handle loops in 3D easily
- Compact (only 2 angles)
- Featherstone's algorithms can be used to solve more complicated systems.

Today's lecture focuses on **Maximal Coordinates**

Acrobot Diagram with Maximal Coordinates:



$$Q = [x_0, y_0, x_1, y_1, x_2, y_2]$$

Mass points in are considered in isolation, in this example all points will have the same mass.

$$Q = [x_0, y_0, x_1, y_1, x_2, y_2]$$

$$M = \begin{bmatrix} m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & m \end{bmatrix}; \text{ M's diagonal}$$

$$q'' = [f_{01x}, f_{01y}, f_{11x}, f_{11y}, f_{21x}, f_{21y}]$$

$$M * q'' = f(t)$$

Constraints:

Length of rod = l

The Constraint Function: $C(q) = 0$; $C(q) = [C_1(q), C_2(q), C_3(q), C_4(q)]$

- $X_0 = 0$
- $Y_0 = 0$
- $x_1^2 + y_1^2 = l^2$
- $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$

So,

- $C_1(q) = x_0$
- $C_2(q) = y_0$
- $C_3(q) = x_1^2 + y_1^2 - l^2$
- $C_4(q) = (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2$

of constraints < # of degrees of freedom.

The new model becomes:

$Mq'' = f(t) + f_c$; where f_c is the constraint force and f_c can not alter the energy in the system. The system moves only from external forces.

$$C(q) = 0$$

Manifold Diagram:

Each set of constraints maps to a position on the manifold. f_c must always be perpendicular to the tangent plane at point q on the manifold so that the dot product of f_c and the derivative is 0. Representing 0 net change in work for the system.

The normal space is spanned by the rows of dC/dq a 4x6 matrix

$$f_c = \left(\frac{dC}{dq}\right)^T * \lambda ; \lambda \in R^4 \text{ called a lagrange multiplier}$$

Our main equation is now

$$(1) Mq'' = f(t) + \left(\frac{dC}{dq}\right)^T * \lambda$$

$$(2) C(q) = 0$$

To solve, differentiate $C(q)$ with respect to time.

$$0 = \frac{d}{dt} * C(q) = \left(\frac{dC}{dq}\right) * q'$$

$$0 = \frac{dC}{dq} * q'' + \left(\frac{d}{dt}\right) * \left(\frac{dC}{dq}\right) * q'$$

By factoring: $\frac{d}{dq} * (dC/dt) = dC'/dq$

Continuing, we can write out our main equation while inverting lambda:

$$Mq'' + (dC/dq)^T * \lambda = f(t) : 6 \text{ equations}$$

$$\frac{dC}{dq} * q'' = -(dC'/dq) * q' : 4 \text{ equations}$$

$$\begin{bmatrix} M & \left(\frac{dC}{dq}\right)^T \\ dC/dq & 0 \end{bmatrix} * \begin{bmatrix} q'' \\ \lambda \end{bmatrix} = \begin{bmatrix} f(t) \\ -\left(\frac{dC'}{dq}\right) * q' \end{bmatrix}$$

$$Mq'' = f(t) + f_c$$

Problems with the simulation

Constant drift because of numerical simulator and only real requirement is $C''=0$ to stabilize a location on the manifold.

The **baumgarte stabilization** is used to correct the simulation.

$$C'' + \alpha C' + \beta C = 0$$

Revised Equation:

$$\begin{bmatrix} M & \left(\frac{dC}{dq}\right)^T \\ dC/dq & 0 \end{bmatrix} * \begin{bmatrix} q'' \\ \lambda \end{bmatrix} = \begin{bmatrix} f(t) \\ -\left(\frac{dC}{dq}\right) * q' - \alpha \left(\frac{dC}{dq}\right) * q' - \beta C \end{bmatrix}$$