Constraints

Examples of constraints:

Hinge joints, anatomical joints like a shoulder, elbow, or wrist.

Different joints have different degrees of freedom and have different limitations on that freedom.

Consider a model with two or more bodies connected by joints. The bodies can also be represented in an acyclic graph or in more complicated hierarchies a cyclic one.

Diagram 1:

Graph 1:
Diagram 2:

Simple Graphs:

Chain system

Diagram 3:

More complicated system with a loop.

Graph 3:

From last lecture regarding Lagrange Dynamics

\[ M(q) \cdot q'' = f(q, q', t) \]

\[ Q = [\alpha, \beta] \]
In this situation \( M(q) \cdot q'' = f(q, q', t) \) but it is much more complex.

This is referred to as the \textit{minimal coordinate} or \textit{reduced coordinate} approach.

Features of this approach:

- Complex mathematics
- Can't handle loops in 3D easily
- Compact (only 2 angles)
- Featherstone’s algorithms can be used to solve more complicated systems.

\textbf{Today’s lecture focuses on \textit{Maximal Coordinates}}
Acrobot Diagram with Maximal Coordinates:

\[ Q = [x_0, y_0, x_1, y_1, x_2, y_2] \]

Mass points in are considered in isolation, in this example all points will have the same mass.

\[ Q = [x_0, y_0, x_1, y_1, x_2, y_2] \]

\[ M = \begin{bmatrix} m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m \end{bmatrix}; \text{M's diagonal} \]

\[ q'' = [f01x, f01y, f11x, f11y, f21x, f21y] \]

\[ M \cdot q'' = f(t) \]

**Constraints:**

*Length of rod = l*

The Constraint Function: \( C(q) = 0; C(q) = [C1(q), C2(q), C3(q), C4(q)] \)

- \( X0 = 0 \)
- \( Y0 = 0 \)
- \( x1^2 + y1^2 = l^2 \)
- \( (x2 - x1)^2 + (y2 - y1)^2 = l^2 \)

So,

- \( C1(q) = x0 \)
- \( C2(q) = y0 \)
- \( C3(q) = x1^2 + y1^2 - l^2 \)
- \( C4(q) = (x2 - x1)^2 + (y2 - y1)^2 - l^2 \)

\# of constraints < \# of degrees of freedom.
The new model becomes:

\[ M q'' = f(t) + f_c \]; where \( f_c \) is the constraint force and \( f_c \) can not alter the energy in the system. The system moves only from external forces.

\[ C(q) = 0 \]

**Manifold Diagram:**

Each set of constraints maps to a position on the manifold. \( f_c \) must always be perpendicular to the tangent plane at point \( q \) on the manifold so that the dot product of \( f_c \) and the derivative is 0. Representing 0 net change in work for the system.

The normal space is spanned by the rows of \( dC/dq \) a 4x6 matrix

\[ f_c = \left( \frac{dc}{dq} \right)^T \lambda; \lambda \in R^4 \text{ called a lagrange multiplier} \]

Our main equation is now

\[
\begin{align*}
(1) \quad M q'' &= f(t) + \left( \frac{dc}{dq} \right)^T \lambda \\
(2) \quad C(q) &= 0
\end{align*}
\]

To solve, differentiate \( C(q) \) with respect to time.

\[ 0 = \frac{d}{dt} C(q) = \left( \frac{dc}{dq} \right) * q' \]

\[ 0 = \frac{dC}{dq} * q'' + \left( \frac{d}{dt} \right) \left( \frac{dc}{dq} \right) * q' \]

By factoring: \( \frac{d}{dq} \left( dC/dq \right) = dC'/dq \)

Continuing, we can write out our main equation while inverting lambda:

\[ M q'' + \left( \frac{dc}{dq} \right)^T \lambda = f(t) : 6 \text{ equations} \]

\[ \frac{dc}{dq} * q'' = -(dC'/dq) * q' : 4 \text{ equations} \]

\[
\begin{bmatrix}
M \\
\frac{dc}{dq}
\end{bmatrix}
\begin{bmatrix}
\frac{dc}{dq}^T \\
0
\end{bmatrix}
\begin{bmatrix}
q'' \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f(t) \\
- \left( \frac{dc}{dq} \right) * q'
\end{bmatrix}
\]

\[ M q'' = f(t) + f_c \]
Problems with the simulation

Constant drift because of numerical simulator and only real requirement is $C''=0$ to stabilize a location on the manifold.

The Baumgarte stabilization is used to correct the simulation.

$$C'' + \alpha C' + \beta C = 0$$

Revised Equation:

$$\begin{bmatrix}
    M & \left(\frac{dc}{dq}\right)^T \\
    \frac{dc}{dq} & 0
\end{bmatrix}
\begin{bmatrix}
    q'' \\
    \lambda
\end{bmatrix}
= 
\begin{bmatrix}
    f(t) \\
    -\left(\frac{dc}{dq}\right) q' - \alpha \left(\frac{dc}{dq}\right) q' - \beta C
\end{bmatrix}$$