Geometric Queries for Ray Tracing

Ray-Surface Intersection
Barycentric Coordinates
[Angel Ch. 11]
Ray-Surface Intersections

• Necessary in ray tracing
• General implicit surfaces
• General parametric surfaces
• Specialized analysis for special surfaces
  – Spheres
  – Planes
  – Polygons
  – Quadrics
Intersection of Rays and Parametric Surfaces

- Ray in parametric form
  - Origin \( \mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T \)
  - Direction \( \mathbf{d} = [x_d \ y_d \ z_d]^T \)
  - Assume \( \mathbf{d} \) is normalized \( (x_d^2 + y_d^2 + z_d^2 = 1) \)
  - Ray \( \mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} \ t \) for \( t > 0 \)

- Surface in parametric form
  - Point \( \mathbf{q} = g(u, v) \), possible bounds on \( u, v \)
  - Solve \( \mathbf{p}_0 + \mathbf{d} \ t = g(u, v) \)
  - Three equations in three unknowns \( (t, u, v) \)
Intersection of Rays and Implicit Surfaces

- Ray in parametric form
  - Origin \( p_0 = [x_0 \ y_0 \ z_0]^T \)
  - Direction \( d = [x_d \ y_d \ z_d]^T \)
  - Assume \( d \) normalized \((x_d^2 + y_d^2 + z_d^2 = 1)\)
  - Ray \( p(t) = p_0 + d \ t \) for \( t > 0 \)

- Implicit surface
  - Given by \( f(q) = 0 \)
  - Consists of all points \( q \) such that \( f(q) = 0 \)
  - Substitute ray equation for \( q \): \( f(p_0 + d \ t) = 0 \)
  - Solve for \( t \) (univariate root finding)
  - Closed form (if possible), otherwise numerical approximation
Ray-Sphere Intersection I

• Common and easy case
• Define sphere by
  – Center $\mathbf{c} = [x_c \ y_c \ z_c]^T$
  – Radius $r$
  – Surface $f(q) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$
• Plug in ray equations for $x, y, z$:

$$x = x_0 + x_d t, \quad y = y_0 + y_d t, \quad z = z_0 + z_d t$$

• And we obtain a scalar equation for $t$:

$$(x_0 + x_d t - x_c)^2 + (y_0 + y_d t - y_c)^2 + (z_0 + z_d t - z_c)^2 = r^2$$
Ray-Sphere Intersection II

• Simplify to

\[ at^2 + bt + c = 0 \]

where

\[ a = x_d^2 + y_d^2 + z_d^2 = 1 \quad \text{since} \quad |d| = 1 \]
\[ b = 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c)) \]
\[ c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2 \]

• Solve to obtain \( t_0 \) and \( t_1 \)

\[ t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \]

Check if \( t_0, t_1 > 0 \) (ray)
Return \( \min(t_0, t_1) \)
Ray-Sphere Intersection III

• For lighting, calculate unit normal

\[ n = \frac{1}{r} \begin{bmatrix} (x_i - x_c) & (y_i - y_c) & (z_i - z_c) \end{bmatrix}^T \]

• Negate if ray originates inside the sphere!
• Note possible problems with roundoff errors
Simple Optimizations

• Factor common subexpressions

• Compute only what is necessary
  – Calculate $b^2 - 4c$, abort if negative
  – Compute normal only for closest intersection
  – Other similar optimizations
Ray-Quadric Intersection

- Quadric $f(p) = f(x, y, z) = 0$, where $f$ is polynomial of order 2
- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder
- Closed form solution as for sphere
- Important case for modelling in ray tracing
- Combine with CSG
Ray-Polygon Intersection I

• Assume planar polygon in 3D
  1. Intersect ray with plane containing polygon
  2. Check if intersection point is inside polygon

• Plane
  – Implicit form: $ax + by + cz + d = 0$
  – Unit normal: $\mathbf{n} = [a \ b \ c]^T$ with $a^2 + b^2 + c^2 = 1$

• Substitute:

$$a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0$$

• Solve:

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}$$
Ray-Polygon Intersection II

- Substitute $t$ to obtain intersection point in plane

- Rewrite using dot product

\[
t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(n \cdot p_0 + d)}{n \cdot d}
\]

- If $n \cdot d = 0$, no intersection (ray parallel to plane)

- If $t \leq 0$, the intersection is behind ray origin
Test if point inside polygon

- Could use even-odd rule, or winding rule
- Easier if polygon is in 2D (project from 3D to 2D)
- Easier for triangles (tessellate polygons)
Point-in-triangle testing

• Critical for polygonal models

• Project the triangle, and point of plane intersection, onto one of the planes $x = 0$, $y = 0$, or $z = 0$ (pick a plane not perpendicular to triangle) (such a choice always exists)

• Then, do the 2D test in the plane, by computing barycentric coordinates (follows next)
Outline

• Ray-Surface Intersections
• Special cases: sphere, polygon
• Barycentric Coordinates
Interpolated Shading for Ray Tracing

- Assume we know normals at vertices
- How do we compute normal of interior point?
- Need linear interpolation between 3 points
- Barycentric coordinates
- Yields same answer as scan conversion
Barycentric Coordinates in 1D

• Linear interpolation
  – \( p(t) = (1 - t)p_1 + t p_2, \ 0 \leq t \leq 1 \)
  – \( p(t) = \alpha \ p_1 + \beta \ p_2 \) where \( \alpha + \beta = 1 \)
  – \( p \) is between \( p_1 \) and \( p_2 \) iff \( 0 \leq \alpha, \beta \leq 1 \)

• Geometric intuition
  – Weigh each vertex by ratio of distances from ends

\[ p_1 \quad p \quad p_2 \]

• \( \alpha, \beta \) are called barycentric coordinates
Barycentric Coordinates in 2D

• Now, we have 3 points instead of 2

Define 3 barycentric coordinates, $\alpha$, $\beta$, $\gamma$

$p = \alpha \ p_1 + \beta \ p_2 + \gamma \ p_3$

$p$ inside triangle iff $0 \leq \alpha, \beta, \gamma \leq 1, \alpha + \beta + \gamma = 1$

• How do we calculate $\alpha, \beta, \gamma$ given $p$?
Barycentric Coordinates for Triangle

- Coordinates are ratios of triangle areas

\[
\alpha = \frac{\text{Area}(C_0C_1C_2)}{\text{Area}(C_0C_1C_2)} \\
\beta = \frac{\text{Area}(C_0C_2C)}{\text{Area}(C_0C_1C_2)} \\
\gamma = \frac{\text{Area}(C_0C_1C)}{\text{Area}(C_0C_1C_2)} = 1 - \alpha - \beta
\]

- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.
Negative Area

Point C is outside of the triangle!

\[ \alpha = \frac{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} > 0 \]

\[ \beta = \frac{\text{Area}(\text{C}_0\text{C}_2\text{C}_1)}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} > 0 \]

\[ \gamma = \frac{\text{Area}(\text{C}_0\text{C}_1\text{C})}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} < 0 \]
Computing Triangle Area in 3D

- Use cross product
- Parallelogram formula
- \( \text{Area}(ABC) = \frac{1}{2} |(B - A) \times (C - A)| \)
- How to get correct sign for barycentric coordinates?
  - tricky, but possible:
    - compare directions of vectors \((B - A) \times (C - A)\), for triangles \(CC_1C_2\) vs \(C_0C_1C_2\), etc.
    - (either 0 (sign+) or 180 deg (sign-) angle)
  - easier alternative: project to 2D, use 2D formula
  - projection to 2D preserves barycentric coordinates
Computing Triangle Area in 2D

• Suppose we project the triangle to xy plane

• Area(xy-projection(ABC)) =

  \[(1/2) \left( (b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y) \right)\]

• This formula gives correct sign (important for barycentric coordinates)
Summary

• Ray-Surface Intersections
• Special cases: sphere, polygon
• Barycentric Coordinates