

Geometric Queries for Ray Tracing

Ray-Surface Intersection
Barycentric Coordinates
[Angel Ch. 11]

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Ray-Surface Intersections

- Necessary in ray tracing
- General implicit surfaces
- General parametric surfaces
- Specialized analysis for special surfaces
 - Spheres
 - Planes
 - Polygons
 - Quadrics

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Intersection of Rays and Parametric Surfaces

- Ray in parametric form
 - Origin $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
 - Direction $\mathbf{d} = [x_d \ y_d \ z_d]^T$
 - Assume \mathbf{d} is normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
 - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} t$ for $t > 0$
- Surface in parametric form
 - Point $\mathbf{q} = \mathbf{g}(u, v)$, possible bounds on u, v
 - Solve $\mathbf{p}_0 + \mathbf{d} t = \mathbf{g}(u, v)$
 - Three equations in three unknowns (t, u, v)

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Intersection of Rays and Implicit Surfaces

- Ray in parametric form
 - Origin $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
 - Direction $\mathbf{d} = [x_d \ y_d \ z_d]^T$
 - Assume \mathbf{d} normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
 - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} t$ for $t > 0$
- Implicit surface
 - Given by $f(\mathbf{q}) = 0$
 - Consists of all points \mathbf{q} such that $f(\mathbf{q}) = 0$
 - Substitute ray equation for \mathbf{q} : $f(\mathbf{p}_0 + \mathbf{d} t) = 0$
 - Solve for t (univariate root finding)
 - Closed form (if possible), otherwise numerical approximation

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Ray-Sphere Intersection I

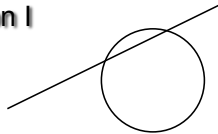
- Common and easy case
- Define sphere by
 - Center $\mathbf{c} = [x_c \ y_c \ z_c]^T$
 - Radius r
 - Surface $f(\mathbf{q}) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$
- Plug in ray equations for x, y, z :

$$x = x_0 + x_d t, \quad y = y_0 + y_d t, \quad z = z_0 + z_d t$$

- And we obtain a scalar equation for t :

$$(x_0 + x_d t - x_c)^2 + (y_0 + y_d t - y_c)^2 + (z_0 + z_d t - z_c)^2 = r^2$$

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Ray-Sphere Intersection II

- Simplify to

$$at^2 + bt + c = 0$$

where

$$a = x_d^2 + y_d^2 + z_d^2 = 1 \quad \text{since } |d| = 1$$

$$b = 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c))$$

$$c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2$$

- Solve to obtain t_0 and t_1

$$t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \quad \text{Check if } t_0, t_1 > 0 \text{ (ray)} \\ \text{Return } \min(t_0, t_1)$$

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Ray-Sphere Intersection III

- For lighting, calculate unit normal

$$n = \frac{1}{r} [(x_i - x_c) \quad (y_i - y_c) \quad (z_i - z_c)]^T$$

- Negate if ray originates inside the sphere!
- Note possible problems with roundoff errors

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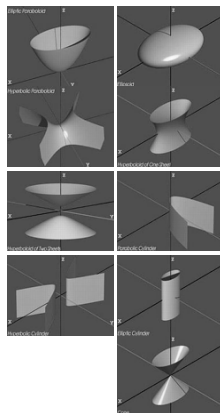
Simple Optimizations

- Factor common subexpressions
- Compute only what is necessary
 - Calculate $b^2 - 4c$, abort if negative
 - Compute normal only for closest intersection
 - Other similar optimizations

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Ray-Quadric Intersection

- Quadric $f(\mathbf{p}) = f(x, y, z) = 0$, where f is polynomial of order 2
- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder
- Closed form solution as for sphere
- Important case for modelling in ray tracing
- Combine with CSG



Ray-Polygon Intersection I

- Assume planar polygon in 3D
 1. Intersect ray with plane containing polygon
 2. Check if intersection point is inside polygon
- Plane
 - Implicit form: $ax + by + cz + d = 0$
 - Unit normal: $\mathbf{n} = [a \quad b \quad c]^T$ with $a^2 + b^2 + c^2 = 1$
- Substitute:

$$a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0$$

- Solve:

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}$$

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Ray-Polygon Intersection II

- Substitute t to obtain intersection point in plane
- Rewrite using dot product

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(\mathbf{n} \cdot \mathbf{p}_0 + d)}{\mathbf{n} \cdot \mathbf{d}}$$
- If $\mathbf{n} \cdot \mathbf{d} = 0$, no intersection (ray parallel to plane)
- If $t \leq 0$, the intersection is behind ray origin

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Test if point inside polygon

- Use even-odd rule or winding rule
- Easier if polygon is in 2D (project from 3D to 2D)
- Easier for triangles (tessellate polygons)

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Point-in-triangle testing

- Critical for polygonal models
- Project the triangle, and point of plane intersection, onto one of the planes $x = 0$, $y = 0$, or $z = 0$ (pick a plane not perpendicular to triangle) (such a choice always exists)
- Then, do the 2D test in the plane, by computing barycentric coordinates (follows next)

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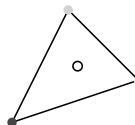
Outline

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates

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Interpolated Shading for Ray Tracing

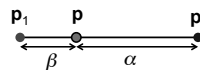
- Assume we know normals at vertices
- How do we compute normal of interior point?
- Need linear interpolation between 3 points
- Barycentric coordinates
- Yields same answer as scan conversion



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Barycentric Coordinates in 1D

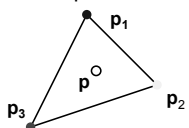
- Linear interpolation
 - $p(t) = (1 - t)p_1 + t p_2, 0 \leq t \leq 1$
 - $p(t) = \alpha p_1 + \beta p_2$ where $\alpha + \beta = 1$
 - p is between p_1 and p_2 iff $0 \leq \alpha, \beta \leq 1$
- Geometric intuition
 - Weigh each vertex by ratio of distances from ends



- α, β are called barycentric coordinates

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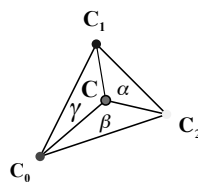
Barycentric Coordinates in 2D

- Now, we have 3 points instead of 2
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- A diagram showing a triangle with vertices labeled p_1 , p_2 , and p_3 . A point p is located inside the triangle.
- Define 3 barycentric coordinates, α, β, γ
 - $p = \alpha p_1 + \beta p_2 + \gamma p_3$
 - p inside triangle iff $0 \leq \alpha, \beta, \gamma \leq 1, \alpha + \beta + \gamma = 1$
 - How do we calculate α, β, γ given p ?

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Barycentric Coordinates for Triangle

- Coordinates are ratios of triangle areas



$$\alpha = \frac{\text{Area}(CC_1C_2)}{\text{Area}(C_0C_1C_2)}$$

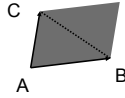
$$\beta = \frac{\text{Area}(C_0CC_2)}{\text{Area}(C_0C_1C_2)}$$

$$\gamma = \frac{\text{Area}(C_0C_1C)}{\text{Area}(C_0C_1C_2)} = 1 - \alpha - \beta$$

- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.

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Computing Triangle Area in 3D



- Use cross product
- Parallelogram formula
- $\text{Area}(ABC) = (1/2) |(B - A) \times (C - A)|$
- How to get correct sign for barycentric coordinates?
 - tricky, but possible:
 - compare directions of vectors $(B - A) \times (C - A)$, for triangles CC_1C_2 vs $C_0C_1C_2$, etc. (either 0 (sign+) or 180 deg (sign-) angle)
 - easier alternative: project to 2D, use 2D formula
 - projection to 2D preserves barycentric coordinates

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Computing Triangle Area in 2D

- Suppose we project the triangle to xy plane
- $\text{Area}(xy\text{-projection}(ABC)) =$
 $(1/2) ((b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y))$
- This formula gives correct sign (important for barycentric coordinates)

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Summary

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates

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