Quaternions and Rotations

Rotations
Quaternions
Motion Capture
[Angel Ch. 3.14]
Rotations

• Very important in computer animation and robotics

• Joint angles, rigid body orientations, camera parameters

• 2D or 3D
Rotations in Three Dimensions

• Orthogonal matrices:

\[ RR^T = R^T R = I \]
\[ \det(R) = 1 \]
Representing Rotations in 3D

• Rotations in 3D have essentially three parameters

• Axis + angle (2 DOFs + 1 DOFs)
  – How to represent the axis?
    Longitude / latitude have singularities

• 3x3 matrix
  – 9 entries (redundant)
Representing Rotations in 3D

- **Euler angles**
  - roll, pitch, yaw
  - no redundancy (good)
  - gimbal lock singularities

- **Quaternions**
  - generally considered the “best” representation
  - redundant (4 values), but only by one DOF (not severe)
  - stable interpolations of rotations possible

Euler Angles

1. Yaw
   rotate around y-axis

2. Pitch
   rotate around (rotated) x-axis

3. Roll
   rotate around (rotated) y-axis

Gimbal Lock

When all three gimbals are lined up (in the same plane), the system can only move in two dimensions from this configuration, not three, and is in *gimbal lock*.

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Outline

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• Motion Capture
Quaternions

• Generalization of complex numbers

• Three imaginary numbers: $i, j, k$

\[ i^2 = -1, \ j^2 = -1, \ k^2 = -1, \]
\[ ij = k, \ jk = i, \ ki = j, \ ji = -k, \ kj = -i, \ ik = -j \]

• $q = s + x \ i + y \ j + z \ k$, \hspace{1cm} s,x,y,z are scalars
Quaternions

• Invented by Hamilton in 1843 in Dublin, Ireland

• Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

\[ i^2 = j^2 = k^2 = i j k = -1 \]

& cut it on a stone of this bridge.

Quaternions

- Quaternions are **not** commutative!

\[ q_1 q_2 \neq q_2 q_1 \]

- However, the following hold:

\[
(q_1 q_2) q_3 = q_1 (q_2 q_3) \\
(q_1 + q_2) q_3 = q_1 q_3 + q_2 q_3 \\
q_1 (q_2 + q_3) = q_1 q_2 + q_1 q_3 \\
\alpha (q_1 + q_2) = \alpha q_1 + \alpha q_2 \quad (\alpha \text{ is scalar}) \\
(\alpha q_1) q_2 = \alpha (q_1 q_2) = q_1 (\alpha q_2) \quad (\alpha \text{ is scalar})
\]

- I.e. all usual manipulations are valid, except cannot reverse multiplication order.
Quaternions

• Exercise: multiply two quaternions

$$(2 - i + j + 3k) (-1 + i + 4j - 2k) = ...$$
Quaternion Properties

- \( q = s + x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \)
- Norm: \( |q|^2 = s^2 + x^2 + y^2 + z^2 \)
- Conjugate quaternion: \( \bar{q} = s - x \mathbf{i} - y \mathbf{j} - z \mathbf{k} \)
- Inverse quaternion: \( q^{-1} = \bar{q} / |q|^2 \)
- Unit quaternion: \( |q| = 1 \)
- Inverse of unit quaternion: \( q^{-1} = \bar{q} \)
Rotations are represented by *unit* quaternions

\[ q = s + x i + y j + z k \]

\[ s^2 + x^2 + y^2 + z^2 = 1 \]

Unit quaternion sphere (unit sphere in 4D)

Source: Wolfram Research

unit sphere in 4D
Rotations to Unit Quaternions

• Let (unit) rotation axis be \([u_x, u_y, u_z]\), and angle \(\theta\)

• Corresponding quaternion is

\[
q = \cos(\theta/2) + \\
\sin(\theta/2) u_x \mathbf{i} + \sin(\theta/2) u_y \mathbf{j} + \sin(\theta/2) u_z \mathbf{k}
\]

• Composition of rotations \(q_1\) and \(q_2\) equals \(q = q_2 q_1\)

• 3D rotations do not commute!
Unit Quaternions to Rotations

• Let \( v \) be a (3-dim) vector and let \( q \) be a unit quaternion

• Then, the corresponding rotation transforms vector \( v \) to \( q \, v \, q^{-1} \)

(\( v \) is a quaternion with scalar part equaling 0, and vector part equaling \( v \))

For \( q = a + b \imath + c \jmath + d \kappa \)

\[
R = \begin{pmatrix}
    a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\
    2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\
    2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2
\end{pmatrix}
\]
Quaternions

- Quaternions $q$ and $-q$ give the same rotation!

- Other than this, the relationship between rotations and quaternions is unique
Quaternion Interpolation

• Better results than Euler angles
• A quaternion is a point on the 4-D unit sphere
  – interpolating rotations requires a unit quaternion at each step -- another point on the 4-D sphere
  – move with constant angular velocity along the great circle between the two points
  – Spherical Linear intERPolation (SLERPing)
• Any rotation is given by 2 quaternions, so pick the shortest SLERP
Quaternions Interpolation

- To interpolate more than two points:
  - solve a non-linear variational constrained optimization (numerically)

- Further information: Ken Shoemake in the SIGGRAPH '85 proceedings (Computer Graphics, V. 19, No. 3, P.245)
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What is Motion Capture?

• Motion capture is the process of tracking real-life motion in 3D and recording it for use in any number of applications.
Why Motion Capture?

• Keyframes are generated by instruments measuring a human performer — they do not need to be set manually
• The details of human motion such as style, mood, and shifts of weight are reproduced with little effort
Mocap Technologies: Optical

- Multiple high-resolution, high-speed cameras
- Light bounced from camera off of reflective markers
- High quality data
- Markers placeable anywhere
- Lots of work to extract joint angles
- Occlusion
- Which marker is which? (correspondence problem)
- 120-240 Hz @ 1Megapixel
Facial Motion Capture
Mocap Technologies: Electromagnetic

- Sensors give both position and orientation
- No occlusion or correspondence problem
- Little post-processing
- Limited accuracy
Mocap Technologies: Exoskeleton

- Really Fast (~500Hz)
- No occlusion or correspondence problem
- Little error
- Movement restricted
- Fixed sensors
Motion Capture

• Why not?
  – Difficult for non-human characters
    • Can you move like a hamster / duck / eagle ?
    • Can you capture a hamster’s motion?
  – Actors needed
    • Which is more economical:
      – Paying an animator to place keys
      – Hiring a Martial Arts Expert
When to use Motion Capture?

• Complicated character motion
  – Where “uncomplicated” ends and “complicated” begins is up to question
  – A walk cycle is often more easily done by hand
  – A Flying Monkey Kick might be worth the overhead of mocap

• Can an actor better express character personality than the animator?
Summary

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