Geometric Queries for Ray Tracing

Ray-Surface Intersection
Barycentric Coordinates
[Angel Ch. 11]
Ray-Surface Intersections

- Necessary in ray tracing
- General implicit surfaces
- General parametric surfaces
- Specialized analysis for special surfaces
  - Spheres
  - Planes
  - Polygons
  - Quadrics
Intersection of Rays and Parametric Surfaces

• Ray in parametric form
  – Origin \( p_0 = [x_0, y_0, z_0]^T \)
  – Direction \( d = [x_d, y_d, z_d]^T \)
  – Assume \( d \) is normalized \( (x_d^2 + y_d^2 + z_d^2 = 1) \)
  – Ray \( p(t) = p_0 + d \ t \) for \( t > 0 \)

• Surface in parametric form
  – Point \( q = g(u, v) \), possible bounds on \( u, v \)
  – Solve \( p_0 + d \ t = g(u, v) \)
  – Three equations in three unknowns \( (t, u, v) \)
Intersection of Rays and Implicit Surfaces

• Ray in parametric form
  – Origin \( p_0 = [x_0 \ y_0 \ z_0]^T \)
  – Direction \( d = [x_d \ y_d \ z_d]^T \)
  – Assume \( d \) normalized (\( x_d^2 + y_d^2 + z_d^2 = 1 \))
  – Ray \( p(t) = p_0 + d \ t \) for \( t > 0 \)

• Implicit surface
  – Given by \( f(q) = 0 \)
  – Consists of all points \( q \) such that \( f(q) = 0 \)
  – Substitute ray equation for \( q \): \( f(p_0 + d \ t) = 0 \)
  – Solve for \( t \) (univariate root finding)
  – Closed form (if possible), otherwise numerical approximation
Ray-Sphere Intersection I

- Common and easy case
- Define sphere by
  - Center \( \mathbf{c} = [x_c, y_c, z_c]^T \)
  - Radius \( r \)
  - Surface \( f(q) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0 \)
- Plug in ray equations for \( x, y, z \):
  \[
  x = x_0 + x_d t, \quad y = y_0 + y_d t, \quad z = z_0 + z_d t
  \]
- And we obtain a scalar equation for \( t \):
  \[
  (x_0 + x_d t - x_c)^2 + (y_0 + y_d t - y_c)^2 + (z_0 + z_d t - z_c)^2 = r^2
  \]
Ray-Sphere Intersection II

- Simplify to

\[ at^2 + bt + c = 0 \]

where

\[
\begin{align*}
    a &= x_d^2 + y_d^2 + z_d^2 = 1 & \text{since } |d| = 1 \\
    b &= 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c)) \\
    c &= (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2
\end{align*}
\]

- Solve to obtain \( t_0 \) and \( t_1 \)

\[
    t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}
\]

Check if \( t_0, t_1 > 0 \) (ray)  
Return \( \min(t_0, t_1) \)
Ray-Sphere Intersection III

• For lighting, calculate unit normal

\[ n = \frac{1}{r} [(x_i - x_c) \ (y_i - y_c) \ (z_i - z_c)]^T \]

• Negate if ray originates inside the sphere!
• Note possible problems with roundoff errors
Simple Optimizations

- Factor common subexpressions

- Compute only what is necessary
  - Calculate $b^2 - 4c$, abort if negative
  - Compute normal only for closest intersection
  - Other similar optimizations
Ray-Quadric Intersection

- Quadric $f(p) = f(x, y, z) = 0$, where $f$ is polynomial of order 2
- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder
- Closed form solution as for sphere
- Important case for modelling in ray tracing
- Combine with CSG
Ray-Polygon Intersection I

- Assume planar polygon in 3D
  1. Intersect ray with plane containing polygon
  2. Check if intersection point is inside polygon

- Plane
  - Implicit form: $ax + by + cz + d = 0$
  - Unit normal: $\mathbf{n} = [a \ b \ c]^T$ with $a^2 + b^2 + c^2 = 1$

- Substitute:
  $$a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0$$

- Solve:
  $$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}$$
Ray-Polygon Intersection II

- Substitute $t$ to obtain intersection point in plane.

- Rewrite using dot product.

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(n \cdot p_0 + d)}{n \cdot d}$$

- If $n \cdot d = 0$, no intersection (ray parallel to plane).

- If $t \leq 0$, the intersection is behind ray origin.
Test if point inside polygon

- Use even-odd rule or winding rule

- Easier if polygon is in 2D (project from 3D to 2D)

- Easier for triangles (tessellate polygons)
Point-in-triangle testing

• Critical for polygonal models

• Project the triangle, and point of plane intersection, onto one of the planes $x = 0$, $y = 0$, or $z = 0$
  (pick a plane not perpendicular to triangle)
  (such a choice always exists)

• Then, do the 2D test in the plane, by computing barycentric coordinates
  (follows next)
Outline

• Ray-Surface Intersections
• Special cases: sphere, polygon
• Barycentric Coordinates
Interpolated Shading for Ray Tracing

- Assume we know normals at vertices
- How do we compute normal of interior point?
- Need linear interpolation between 3 points
- Barycentric coordinates
- Yields same answer as scan conversion
Barycentric Coordinates in 1D

• Linear interpolation
  – \( p(t) = (1 - t)p_1 + t \ p_2, \ 0 \leq t \leq 1 \)
  – \( p(t) = \alpha \ p_1 + \beta \ p_2 \) where \( \alpha + \beta = 1 \)
  – \( p \) is between \( p_1 \) and \( p_2 \) iff \( 0 \leq \alpha, \ \beta \leq 1 \)

• Geometric intuition
  – Weigh each vertex by ratio of distances from ends

\[ \begin{array}{c}
  p_1 \quad p \quad p_2 \\
  \quad p_2
\end{array} \]

• \( \alpha, \ \beta \) are called barycentric coordinates
Barycentric Coordinates in 2D

- Now, we have 3 points instead of 2
- Define 3 barycentric coordinates, $\alpha$, $\beta$, $\gamma$
- $\mathbf{p} = \alpha \mathbf{p}_1 + \beta \mathbf{p}_2 + \gamma \mathbf{p}_3$
- $\mathbf{p}$ inside triangle iff $0 \leq \alpha, \beta, \gamma \leq 1$, $\alpha + \beta + \gamma = 1$
- How do we calculate $\alpha$, $\beta$, $\gamma$ given $\mathbf{p}$?
Barycentric Coordinates for Triangle

- Coordinates are ratios of triangle areas

\[
\alpha = \frac{\text{Area}(\text{CC}_1\text{C}_2)}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)}
\]

\[
\beta = \frac{\text{Area}(\text{C}_0\text{CC}_2)}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)}
\]

\[
\gamma = \frac{\text{Area}(\text{C}_0\text{C}_1\text{C})}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} = 1 - \alpha - \beta
\]

- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.
Computing Triangle Area in 3D

• Use cross product
• Parallelogram formula
• Area(ABC) = (1/2) |(B – A) x (C – A)|
• How to get correct sign for barycentric coordinates?
  – tricky, but possible:
    compare directions of vectors (B – A) x (C – A), for
    triangles C\textsubscript{1}C\textsubscript{2}C\textsubscript{3} vs C\textsubscript{0}C\textsubscript{1}C\textsubscript{2}, etc.
    (either 0 (sign+) or 180 deg (sign-) angle)
  – easier alternative: project to 2D, use 2D formula
  – projection to 2D preserves barycentric coordinates
Computing Triangle Area in 2D

• Suppose we project the triangle to xy plane

• Area(xy-projection(ABC)) =

\[
\frac{1}{2} \left( (b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y) \right)
\]

• This formula gives correct sign (important for barycentric coordinates)
Summary

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates