Geometric Queries for Ray Tracing

Ray-Surface Intersection
Barycentric Coordinates
[Ch. 13.2 - 13.3]
Ray-Surface Intersections

- Necessary in ray tracing
- General implicit surfaces
- General parametric surfaces
- Specialized analysis for special surfaces
  - Spheres
  - Planes
  - Polygons
  - Quadrics
Intersection of Rays and Parametric Surfaces

• Ray in parametric form
  – Origin $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
  – Direction $\mathbf{d} = [x_d \ y_d \ z_d]^T$
  – Assume $\mathbf{d}$ is normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
  – Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} \ t$ for $t > 0$

• Surface in parametric form
  – Point $\mathbf{q} = g(u, v)$, possible bounds on $u$, $v$
  – Solve $\mathbf{p} + \mathbf{d} \ t = g(u, v)$
  – Three equations in three unknowns $(t, u, v)$
Intersection of Rays and Implicit Surfaces

• Ray in parametric form
  – Origin $p_0 = [x_0 \ y_0 \ z_0]^T$
  – Direction $d = [x_d \ y_d \ z_d]^T$
  – Assume $d$ normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
  – Ray $p(t) = p_0 + d \ t$ for $t > 0$

• Implicit surface
  – Given by $f(q) = 0$
  – Consists of all points $q$ such that $f(q) = 0$
  – Substitute ray equation for $q$: $f(p_0 + d \ t) = 0$
  – Solve for $t$ (univariate root finding)
  – Closed form (if possible), otherwise numerical approximation
Ray-Sphere Intersection I

- Common and easy case
- Define sphere by
  - Center $\mathbf{c} = [x_c, y_c, z_c]^T$
  - Radius $r$
  - Surface $f(q) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$
- Plug in ray equations for $x$, $y$, $z$:
  $$x = x_0 + x_d t, \quad y = y_0 + y_d t, \quad z = z_0 + z_d t$$
- And we obtain a scalar equation for $t$:
  $$\left( x_0 + x_d t - x_c \right)^2 + \left( y_0 + y_d t - y_c \right)^2 + \left( z_0 + z_d t - z_c \right)^2 = r^2$$
Ray-Sphere Intersection II

• Simplify to

\[ at^2 + bt + c = 0 \]

where

\[ a = x_d^2 + y_d^2 + z_d^2 = 1 \quad \text{since } |d| = 1 \]
\[ b = 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c)) \]
\[ c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2 \]

• Solve to obtain \( t_0 \) and \( t_1 \)

\[ t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \]

Check if \( t_0, t_1 > 0 \) (ray)
Return \( \min(t_0, t_1) \)
Ray-Sphere Intersection III

- For lighting, calculate unit normal

\[ n = \frac{1}{r} \begin{bmatrix} (x_i - x_c) & (y_i - y_c) & (z_i - z_c) \end{bmatrix}^T \]

- Negate if ray originates inside the sphere!
- Note possible problems with roundoff errors
Simple Optimizations

• Factor common subexpressions

• Compute only what is necessary
  – Calculate $b^2 - 4c$, abort if negative
  – Compute normal only for closest intersection
  – Other similar optimizations
Ray-Quadric Intersection

- Quadric $f(p) = f(x, y, z) = 0$, where $f$ is polynomial of order 2

- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder

- Closed form solution as for sphere

- Important case for modelling in ray tracing

- Combine with CSG
Ray-Polygon Intersection I

- Assume planar polygon in 3D
  1. Intersect ray with plane containing polygon
  2. Check if intersection point is inside polygon

- Plane
  - Implicit form: \( ax + by + cz + d = 0 \)
  - Unit normal: \( \mathbf{n} = [a, b, c]^T \) with \( a^2 + b^2 + c^2 = 1 \)

- Substitute:
  \[
  a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0
  \]

- Solve:
  \[
  t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}
  \]
Ray-Polygon Intersection II

- Substitute $t$ to obtain intersection point in plane

- Rewrite using dot product

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(n \cdot p_0 + d)}{n \cdot d}$$

- If $n \cdot d = 0$, no intersection (ray parallel to plane)

- If $t \leq 0$, the intersection is behind ray origin
Test if point inside polygon

• Use even-odd rule or winding rule

• Easier if polygon is in 2D (project from 3D to 2D)

• Easier for triangles (tessellate polygons)
Point-in-triangle testing

• Critical for polygonal models

• Project the triangle, and point of plane intersection, onto one of the planes $x = 0, y = 0, \text{or } z = 0$
  (pick a plane not perpendicular to triangle)
  (such a choice always exists)

• Then, do the 2D test in the plane, by computing barycentric coordinates
  (follows next)
Outline

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates
Interpolated Shading for Ray Tracing

• Assume we know normals at vertices
• How do we compute normal of interior point?
• Need linear interpolation between 3 points
• Barycentric coordinates
• Yields same answer as scan conversion
Barycentric Coordinates in 1D

- Linear interpolation
  - \( p(t) = (1 - t)p_1 + t p_2, \) 0 \( \leq \) \( t \leq \) 1
  - \( p(t) = \alpha p_1 + \beta p_2 \) where \( \alpha + \beta = 1 \)
  - \( p \) is between \( p_1 \) and \( p_2 \) iff \( 0 \leq \alpha, \beta \leq 1 \)

- Geometric intuition
  - Weigh each vertex by ratio of distances from ends

- \( \alpha, \beta \) are called barycentric coordinates
Barycentric Coordinates in 2D

- Now, we have 3 points instead of 2
- Define 3 barycentric coordinates, $\alpha$, $\beta$, $\gamma$
- $p = \alpha \ p_1 + \beta \ p_2 + \gamma \ p_3$
- $p$ inside triangle iff $0 \leq \alpha, \beta, \gamma \leq 1$, $\alpha + \beta + \gamma = 1$
- How do we calculate $\alpha, \beta, \gamma$ given $p$?
Barycentric Coordinates for Triangle

- Coordinates are ratios of triangle areas

\[
\alpha = \frac{\text{Area}(\text{CC}_1\text{C}_2)}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} \\
\beta = \frac{\text{Area}(\text{C}_0\text{CC}_2)}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} \\
\gamma = \frac{\text{Area}(\text{C}_0\text{C}_1\text{C})}{\text{Area}(\text{C}_0\text{C}_1\text{C}_2)} = 1 - \alpha - \beta
\]

- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.
Computing Triangle Area in 3D

- Use cross product
- Parallelogram formula
- Area(ABC) = \(\frac{1}{2} \left| \mathbf{(B - A) \times (C - A)} \right|\)
- How to get correct sign for barycentric coordinates?
  - tricky, but possible:
    - compare directions of vectors \((B - A) \times (C - A)\), for triangles \(CC_1C_2\) vs \(C_0C_1C_2\), etc.
    - (either 0 (sign+) or 180 deg (sign-) angle)
  - easier alternative: project to 2D, use 2D formula
  - projection to 2D preserves barycentric coordinates
Computing Triangle Area in 2D

• Suppose we project the triangle to xy plane

• Area(xy-projection(ABC)) =

\[ \frac{1}{2} ((b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y)) \]

• This formula gives correct sign (important for barycentric coordinates)
Summary

• Ray-Surface Intersections
• Special cases: sphere, polygon
• Barycentric Coordinates